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SOME POISSON STRUCTURES ASSOCIATED TO DRINFELD-JIMBO R-MATRICES AND THEIR QUANTIZATION

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ABSTRACT

We describe various ways of constructing Poisson structures associated to Drinfeld–Jimbo R-matrices on homogeneous spaces, research some relations between these structures, and quantize some of them.

There exist two ways to construct Poisson brackets by means of R-matrices on some $\mathfrak{G}$-homogeneous space M. Hereafter we suppose $\mathfrak{G}$ to be a simple Lie group with the Lie algebra $\mathfrak{g}$ over the field $k = \mathbf{R}$ or $\mathbf{C}$. We fix a Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{g}$ and a subset Π of simple roots in the set Ω of all roots of $\mathfrak{g}$ with respect to $\mathfrak{c}$. Then the famous modified Drinfeld–Jimbo R-matrix has the form

$$R = R_{DJ} = \frac{1}{2} \sum_{\alpha\in\Omega_+} X_\alpha \wedge X_{-\alpha} \in \wedge^2\mathfrak{g},$$

where $\{H_\alpha, X_\alpha, X_{-\alpha}\}$, $\alpha \in \Omega_+$, is the Cartan–Chevalley system in $\mathfrak{g}$ and Ω_+ is the set of all positive roots of $\mathfrak{g}$. This R-matrix defines the Sklyanin–Drinfeld

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(SD) bracket $\{,\}_{\rm SD}$ on the Lie group $\mathfrak{G}$ which makes $\mathfrak{G}$ into a Poisson-Lie group (see [D0]).

The first way to construct a Poisson bracket on M is to reduce (in case it is possible) the Sklyanin–Drinfeld bracket. In this case we perceive the algebra of functions on M as a subalgebra of functions on $\mathfrak{G}$. The reduction is possible, if applying the SD bracket to any pair of functions from the subalgebra gives such a function.

Another way is to construct a bilinear operator

$$C^\infty(M) \ni f, g \mapsto \{f,g\}_R = \langle \rho(R), df \otimes dg\rangle \tag{1}$$

where $\rho\colon \mathfrak{g} \to \mathrm{Vect}(M)$ is a representation of $\mathfrak{g}$ in the space of all smooth vector fields on M. Under certain conditions this operator defines a Poisson bracket on M (we call it an R-matrix Poisson bracket). Let us consider these conditions in more details.

Let $x \in M$ be a fixed point. We denote by $\mathfrak{H} = \mathfrak{H}_x = \mathrm{Stab}(x)$ the stabilizer of x in $\mathfrak{G}$ and $\mathfrak{h} = \mathfrak{h}_x \subset \mathfrak{g}$ the corresponding Lie subalgebra. It is proved in [DGM] that the bracket $\{,\}_R$ is a Poisson one if $\mathfrak{h}_x$ contains a maximal nilpotent subalgebra of $\mathfrak{g}$. A similar statement holds for symmetric spaces. We show in this paper that the operator (1) defined by means of $R_{\rm DJ}$ is a Poisson bracket on any symmetric space M.

In case the algebra $C^\infty(M)$ is equipped with another Poisson bracket $\{,\}$ and the range of $\rho(\mathfrak{g})$ lies in the space of Poisson vector fields, i.e. vector fields $X \in \mathrm{Vect}(M)$ preserving the bracket $\{,\}$, then the brackets $\{,\}$ and $\{,\}_R$ are compatible (see for example [DGM]), i.e. each bracket of the family

$$\{,\}_{a,b} = a\{,\} + b\{,\}_R, \quad a, b \in k \tag{2}$$

is Poisson one.

In Section 1 we describe a class of manifolds M which allow us to reduce the Sklyanin–Drinfeld bracket. We show that if M is a Hermitian symmetric space, then the reduced SD bracket coincides with one of the brackets in the family $\{,\}_{a,b}$ with $\{,\}$ being the Kirillov bracket.

Two natural questions arise: whether the reduced SD bracket can be quantized on each space M of the class mentioned above and whether all brackets from the family $\{,\}_{a,b}$ (assuming M to have two Poisson structures as above) can be quantized?

The answer is positive. A new approach to this problem using a nonassociative multiplication as an intermediate step has been developed recently by S. Shnider and the first author and will appear in a future article.

The method of quantizing used in the paper is similar to one used in [DGM] and [DG]. It consists of twisting the initial commutative structure on a $\mathfrak{G}$-homogeneous space M by means of an element $F \in U\mathfrak{g}^{\otimes 2}[[h]]$ (but, unlike [DGM] and [DG], we have to specify this element). Our construction immediately implies that the quantum deformation is flat, i.e. $C^\infty(M)$ does not change under deformation as a vector space. From the algebraic point of view flatness means that the deformed algebra $C^\infty(M)_h$ is a flat $k[[h]]$-module.

1. R-matrix Poisson structures

Let $\mathfrak{g}, \mathfrak{G}, R = R_{\mathrm{DJ}}$ be as above and M a $\mathfrak{G}$-homogeneous space. Consider the element

$$\varphi = [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] \in \wedge^3\mathfrak{g}.$$

Note that the dimension of the subspace of $\mathfrak{g}$-invariant elements in $\wedge^3(\mathfrak{g})$ is equal to one and φ is a generator of the space.

It is obvious that the bracket $\{,\}_R$ constructed according to (1) is a Poisson bracket on M if and only if

$$\mu\langle\rho(\varphi), df \otimes dg \otimes dh\rangle = 0 \tag{3}$$

for arbitrary $f, g, h \in C^\infty(M)$; μ denotes the multiplication in $C^\infty(M)$. Since φ is $\mathfrak{G}$-invariant, it is sufficient to check the equality (3) at some point $x \in M$.

Fix a point $x \in M$. The relation (3) is true if

$$\varphi \in \mathfrak{h} \wedge \mathfrak{g} \wedge \mathfrak{g} \tag{4}$$

where $\mathfrak{h} = \mathfrak{h}_x$, the isotropy subalgebra of x.

PROPOSITION 1.1: *The relation* (4) *holds if a homogeneous space M satisfies one of the following condition:*

1. *$\mathfrak{h}$ contains a maximal nilpotent subalgebra of $\mathfrak{g}$.*
2. *M is a symmetric space, i.e. there exists a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m}$ is a subspace in $\mathfrak{g}$, and an involutive automorphism θ: $\mathfrak{g} \to \mathfrak{g}$ such that* $\theta = \mathrm{id}$ on $\mathfrak{h}$ and $\theta = -\mathrm{id}$ on $\mathfrak{m}$.

Proof: The first case was examined in [DGM]. Consider now the second case. We fix a basis $\{m_i\} \in \mathfrak{m}, \{h_i\} \in \mathfrak{h}$ in $\mathfrak{g}$ and represent φ in this basis. It is obvious that φ does not contain any summands of the type $m_i \otimes m_j \otimes m_k$ since φ is θ-invariant (φ is up to a factor dual to the form $([x,y],z)$, where $x,y,z \in \mathfrak{g}$ and $(,)$ is the Killing form). Therefore, the relation (4) holds, and this completes the proof. ∎

Remark 1.1: Proposition 1.1 is true either over the field $k = \mathbf{R}$ or over the field $k = \mathbf{C}$, but in the second case we mean by M a complex analytic manifold and replace the space $C^\infty(M)$ by the space of all holomorphic functions on M.

In the sequel we denote by $\mathfrak{g}_c$ the compact form of the complexification of the algebra $\mathfrak{g}$ and use the analogous notations for the corresponding group.

It is easy to see that $iR_{\mathrm{DJ}} \in \wedge^2 \mathfrak{g}_c$ and, therefore, iR_{DJ} can be considered as a modified R-matrix on $\mathfrak{g}_c$. Note that this R-matrix was constructed by S. Majid in terms of Manin triples. By the same reason R-matrix iR_{DJ} defines a Poisson bracket on all symmetric spaces of $\mathfrak{G}_c$.

Note that the case of orbits in $\mathfrak{g}_c^*$ is investigated in [KRR], where it has been shown that the operator (1) defines a Poisson bracket on an orbit if and only if it is a symmetric space.

Now we compare the R-matrix brackets with those obtained from the Sklyanin–Drinfeld (SD) brackets by means of Poisson reduction. We call a SD bracket the following one:

$$\{,\}_{\mathrm{SD}} = \{,\}_l - \{,\}_r,$$

where the brackets $\{,\}_l$ (resp. $\{,\}_r$) are defined according to (1) by means of the canonical representation $\rho = \rho_l$ (resp. $\rho = \rho_r$) of $\mathfrak{g}$ in the space of right- (resp. left-) invariant vector fields on $\mathfrak{G}$.

Let M be a $\mathfrak{G}$-homogeneous space. Fix a point $x_0 \in M$. Then there exists a natural embedding

$$C^\infty(M) \to C^\infty(\mathfrak{G}), \quad f_M(x) \mapsto f_{\mathfrak{G}}(g) = f_M(gx_0).$$

Hence, the algebra $C^\infty(M)$ can be considered as a subalgebra of $C^\infty(\mathfrak{G})$.

The following question naturally arises: whether the bracket $\{,\}_{\mathrm{SD}}$ can be reduced to the space $C^\infty(M)$?

The positive answer to this question has been given in [LW] for any orbits of a compact group $\mathfrak{G}_c$ in $\mathfrak{g}_c^*$.

Note that the bracket is reducible if the following relation holds:

$$\rho_r(X)\{f,g\} = 0, \text{ if } X \in \mathfrak{h},\ \rho_r(X)f = \rho_r(X)g = 0. \tag{5}$$

Definition 1.1: Let P be a subset in Π (recall that Π is the fixed set of simple positive roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{c}$). We say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is P-generated if it is generated by the family of elements $\{X_{\pm\alpha}, \alpha \in P\}$ and by a subalgebra of $\mathfrak{c}$. The corresponding homogeneous space is said to be P-generated as well.

Note that if $\mathfrak{h}$ is P-generated, it can be decomposed into a direct sum $\mathfrak{h} = \mathfrak{p}^s \oplus \mathfrak{p}^a$ where the first component is a semisimple Lie algebra generated by elements $\{X_{\pm\alpha}, \alpha \in P\}$ and the second one is an abelian subalgebra of $\mathfrak{c}$.

Of course, the property for a subalgebra $\mathfrak{h}$ to be P-generated depends on choosing the family of simple roots Π. It is easy to see that if $x \in \mathfrak{c}$, then the isotropy algebra of x by *Ad*-action of $\mathfrak{G}$ will be P-generated with respect to some choice of the set of simple roots.

PROPOSITION 1.2: *For any homogeneous P-generated space the relation* (5) *holds and, therefore, the bracket $\{,\}_{\mathrm{SD}}$ can be reduced to the space M.*

Proof: The following relation for $X \in \mathfrak{h}$ implies the statement we need:

$$[\Delta X, R_{\mathrm{DJ}}] \in \mathfrak{h} \wedge \mathfrak{h}. \tag{6}$$

Hereafter Δ denotes the usual coproduct in the enveloping algebra $U\mathfrak{g}$.

Prove now the relation (6). If $X \in \mathfrak{c}$ then it is easy to see that $[\Delta X, R_{\mathrm{DJ}}] = 0$. If $X = X_\beta$, where β is a simple root, then

$$\begin{aligned} 2[\Delta X_\beta, R_{\mathrm{DJ}}] &= [\Delta X_\beta, \sum X_\alpha \wedge X_{-\alpha}] \\ &= \sum (c_{\beta,\alpha} X_{\beta+\alpha} \wedge X_{-\alpha} + c_{\beta,-\alpha} X_\alpha \wedge X_{\beta-\alpha}) \end{aligned} \tag{7}$$

where the sum is taken for all the positive roots α, and we assume X_γ to be equal to zero when γ is not a root. Observe now that in case $\beta + \alpha$ is a root the term $X_{\beta+\alpha} \wedge X_{-\alpha}$ does not appear in (7). Indeed, it could only appear upon multiplying by X_β the terms $X_\alpha \wedge X_{-\alpha}$ and $X_{\beta+\alpha} \wedge X_{-\beta-\alpha}$ from R_{DJ}. Then, it would have the coefficient $c_{\beta,\alpha} + c_{\beta,-\beta-\alpha}$ if $\beta + \alpha$ is a positive root and $c_{\beta,\alpha} - c_{\beta,-\beta-\alpha}$ if $\beta + \alpha$ is a negative root. But in case of a positive root

$c_{\beta,\alpha}+c_{\beta,-\beta-\alpha}=0$, which follows from the relation $c_{\alpha,\beta}=c_{\beta,\gamma}$ when $\alpha+\beta+\gamma=0$ (see [H]). The case when $\beta+\alpha$ is a negative root does not appear at all because α is a positive root and β is a simple root, so that $\beta+\alpha$ cannot be a negative root. It is easy to check that the coefficient by the term $X_\alpha \wedge X_{\beta-\alpha}$ in (7) is equal to zero as well. Thus, only terms of the form $H_\beta \wedge X_\beta \in \mathfrak{h} \wedge \mathfrak{h}$ have a nonzero coefficient. Since the algebra $\mathfrak{p}^s$ is generated by elements $X_{\pm\beta}$ with $\beta \in P$ being simple roots, it is shown that (6) holds. This completes the proof of the proposition. ∎

Now, let us consider an orbit $M \subset \mathfrak{g}^*$ which is a symmetric space. It is equipped with two brackets: the first is the Kirillov bracket, another is the R-matrix one (which coincides with the reduced bracket $\{,\}_l$). Besides that, if M is P-generated, it can be equipped with the invariant reduced bracket $\{,\}_r$.

The following question arises: what is the relation between the reduced bracket $\{,\}_r$ and the Kirillov one?

PROPOSITION 1.3: *If M is an orbit in $\mathfrak{g}^*$ and a P-generated Hermitian symmetric space, then the reduced bracket $\{,\}_r$ coincides (up to a factor) with the Kirillov bracket.*

Proof: It is easy to see that the reduced bracket $\{,\}_r$ is nondegenerate. Hence, $\{,\}_r$ defines by duality a closed invariant differential 2-form which has to be proportional to the corresponding form defined by the Kirillov bracket, because of the following two facts for Hermitian symmetric spaces: the dimension of the space of closed invariant differential i-forms is equal to $\dim H^i(M, \mathbf{C})$, and $\dim H^2(M, \mathbf{C}) = 1$. It completes the proof. ∎

Thus, for a Hermitian symmetric space M we have the family of brackets (2), where $\{,\}$ is the Kirillov bracket (or reduced bracket $\{,\}_r$) and $\{,\}_R$ is the R-matrix bracket (or reduced bracket $\{,\}_l$).

Note that if $M = \mathfrak{G}/\mathfrak{N}$, where $\mathfrak{N} \in \mathfrak{G}$ is the subgroup with Lie algebra $\mathfrak{n}$ generated by $\{X_\alpha, \alpha > 0\}$, then the bracket $\{,\}_{\mathrm{SD}}$ can be reduced to the space M as well and coincides with the R-matrix bracket $\{,\}_l$, because in this case the reduced bracket $\{,\}_r$ is equal to zero.

2. Quantization of reduced SD brackets

It is well-known that the quantum counterpart of a SD structure on a semisimple Lie group is a quantum group. More often the quantum groups are introduced in terms of deformed relations between coordinate functions. But the result of quantization of this structure can be also described in terms of the deformation quantization.

As it is proved in [D2], there exist two series in h, namely $F = F(h) \in U\mathfrak{g}^{\otimes 2}[[h]]$ and $\Phi = \Phi(h) \in U\mathfrak{g}^{\otimes 3}[[h]]$ such that

$$F = 1 \bmod(h), \quad F - F^{21} = R \bmod(h^2), \tag{8}$$

$$F^{12}\Delta^{12}F\Phi = F^{23}\Delta^{23}F \tag{9}$$

and Φ is $\mathfrak{g}$-invariant.

Following [Ta] we introduce the new multiplication in the space $C^\infty(\mathfrak{G})$,

$$f, g \in C^\infty(\mathfrak{G}) \to f * g = \mu F_r^{-1} F_l(f \otimes g), \tag{10}$$

where μ is the usual multiplication and $F_r = \rho_r(F)$ and $F_l = \rho_l(F)$ are the left- and right-invariant bidifferential operators on $\mathfrak{G}$, where we mean under ρ_r and ρ_l the extension up to $U\mathfrak{g}$ of the corresponding representation of $\mathfrak{g}$.

Since Φ is $\mathfrak{g}$-invariant, the 3-differential operators $\Phi_r = \rho_r(\Phi)$ and $\Phi_l = \rho_l(\Phi)$ coincide. Using this fact, the relation (9), and commutativity of F_r and F_l, it is easy to check that the new multiplication will be associative. Due to (8) the relations

$$f * g = fg \bmod(h), \; f * g - g * f = h\{f, g\} \bmod(h^2) \tag{11}$$

with $\{,\} = \{,\}_{\mathrm{SD}}$ are satisfied, which means that the correspondence principle holds, or, in other words, the constructed multiplication quantizes the bracket $\{,\}_{\mathrm{SD}}$.

Consider a P-generated $\mathfrak{G}$-homogeneous manifold M. We want to reduce the multiplication (10) to M in a similar way. Let us show that this can be done using some F.

THEOREM 2.1: *There exist F and Φ satisfying the relations* (8), (9) *and the following condition:*

$$F\Delta(X)F^{-1} \in U\mathfrak{h}^{\otimes 2}[[h]]^0, \quad \textit{if } X \in \mathfrak{h}, \tag{12}$$

where $U\mathfrak{h}^{\otimes 2}[[h]]^0 \subset U\mathfrak{h}^{\otimes 2}[[h]]$ is the maximal ideal generated by $\{X \otimes 1, 1 \otimes X, X \in \mathfrak{h}\}$.

Note that the condition (12) may be perceived as a quantum counterpart of the condition (6).

First, we will prove a lemma. Let $U_h\mathfrak{g}$ be the quantum group corresponding to $\mathfrak{g}$. The algebraic structure on $U_h\mathfrak{g}$ is defined by means of some analytic relations between the generators $\{H_\alpha, X_{\pm\alpha}\}$, $\alpha \in \Pi$ (see [D1]). Let $\mathfrak{h} = \mathfrak{p}^s \oplus \mathfrak{p}^a$ be a P-generated Lie subalgebra of $\mathfrak{g}$. Denote by $U_h\mathfrak{h}$ the "quantum subgroup" corresponding to $\mathfrak{h}$, i.e. the subalgebra of $U_h\mathfrak{g}$ generated by $X_{\pm\alpha}, \alpha \in P$, and $H \in \mathfrak{p}^a$.

LEMMA 2.1: *There exists an algebra isomorphism*

$$i\colon U_h\mathfrak{g} \to U\mathfrak{g}[[h]]$$

such that (a) $i = id$ *on* $\mathfrak{c}$, (b) *being restricted on* $U_h\mathfrak{h}$, i *is an algebra isomorphism between* $U_h\mathfrak{h}$ *and* $U\mathfrak{h}[[h]]$.

Proof: Adding to $\mathfrak{h}$ the elements from $\mathfrak{c}$ we will get the P-generated algebra $\mathfrak{h}'$ which contains the Cartan subalgebra $\mathfrak{c}$. It is easy to see that if the lemma is true for $\mathfrak{h}'$, it will be true for $\mathfrak{h}$. So that we have to prove the lemma in the case when $\mathfrak{h}$ contains the Cartan subalgebra $\mathfrak{c}$. The existence of i with the property (a) has been proved in [D1]. Let $\mathfrak{h} = \mathfrak{p}^s \oplus \mathfrak{p}^a$ be the decomposition, where the first component is a semisimple Lie algebra and the second one is an Abelian subalgebra of $\mathfrak{c}$. Then $\mathfrak{h}$ coincides with

$$(\mathfrak{g})^{\mathfrak{p}^a} = \{X \in \mathfrak{g}; [X, \mathfrak{p}^a] = 0\}. \tag{13}$$

Consider $U_h\mathfrak{p}^s$ as a subalgebra of $U_h\mathfrak{g}$. If i satisfies the condition (a) from the lemma, then

$$i(U_h\mathfrak{p}^s) \subset (U\mathfrak{g})^{\mathfrak{p}^a}[[h]],$$

where

$$(U\mathfrak{g})^{\mathfrak{p}^a} = \{X \in U\mathfrak{g}; [X, \mathfrak{p}^a] = 0\}.$$

On the other hand, there exists an isomorphism

$$i'\colon U_h\mathfrak{p}^s \to U\mathfrak{p}^s[[h]] \subset (U\mathfrak{g})^{\mathfrak{p}^a}[[h]],$$

which is identical mod(h). There exists an inner automorphism of $(U\mathfrak{g})^{\mathfrak{p}^a}[[h]]$ preserving $\mathfrak{c}$ which maps $i(U_h\mathfrak{p}^s)$ onto $i'(U_h\mathfrak{p}^s)$. This follows from the fact that $H^1(\mathfrak{p}^s, (U\mathfrak{g})^{\mathfrak{p}^a}/U\mathfrak{p}^s) = 0$. This proves the assertion (b) of the lemma. ∎

Proof of Theorem 2.1: The algebra $U_h\mathfrak{g}$ has a Hopf algebra structure. Consider the composed map

$$j\colon U\mathfrak{g}[[h]] \xrightarrow{i^{-1}} U_h\mathfrak{g} \xrightarrow{\Delta_h} U_h\mathfrak{g}^{\otimes 2} \xrightarrow{i\otimes i} U\mathfrak{g}[[h]]^{\otimes 2},$$

where Δ_h is the coproduct in $U_h\mathfrak{g}$ and the mapping i is from the previous lemma. This map is an algebra morphism from $U\mathfrak{g}[[h]]$ to $U\mathfrak{g}^{\otimes 2}[[h]]$, which restricts to a morphism from $U\mathfrak{h}[[h]]$ to $U\mathfrak{h}^{\otimes 2}[[h]]$. There exists an invertible element $F \in U\mathfrak{g}^{\otimes 2}[[h]]$ such that $F = 1 \bmod (h)$ and $j(X) = F\Delta(X)F^{-1}$ (this follows from the triviality of cohomology $H^1(\mathfrak{g}, U\mathfrak{g}^{\otimes 2})$, see [D1]). It is clear that this F is as required in the theorem. Now we define Φ using the formula (9). Since the coproduct j is coassociative, Φ has to be a $\mathfrak{g}$-invariant element in $U\mathfrak{g}^{\otimes 3}[[h]]$. The second formula (8) can be checked using the explicit form of the coproduct in $U_h\mathfrak{g}$. The theorem is proved. ∎

Now we show that the multiplication (10) is reducible onto P-generated homogeneous spaces.

THEOREM 2.2: *Let $\mathfrak{G}$ be a simple Lie group with a fixed Cartan subalgebra in its Lie algebra and a set of simple roots Π. Suppose M is a P-generated relative to Π homogeneous space of $\mathfrak{G}$. Then the SD bracket on $\mathfrak{G}$ corresponding to Π can be reduced onto M and the obtained SD bracket on M can be quantized.*

Proof: The fact that the bracket is reducible is the content of Proposition 1.2, and we have to demonstrate that

$$\rho_r(X)\mu F_r^{-1}(f\otimes g) = 0, \quad \text{if } X \in \mathfrak{h} \quad \text{and} \quad \rho_r(X)f = \rho_r(X)g = 0.$$

But this follows from the chain of equalities

$$\begin{aligned}\rho_r(X)\mu F_r^{-1}(f\otimes g) &= \mu\rho_r(\Delta X)F_r^{-1}(f\otimes g)\\ &= \mu F_r^{-1}(F_r\rho(\Delta X)F_r^{-1})(f\otimes g) = 0,\end{aligned}$$

which take place due to Theorem 2.1 and the property of f and g. Therefore, the multiplication (10) is reducible, which completes the proof. ∎

This theorem shows that one can quantize the reduced SD bracket on any P-generated $\mathfrak{G}$-homogeneous manifold M in the sense of deformation quantization.

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